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Fundamental Solutions with Partially Bounded Support

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For certain partial differential operators with constant coefficients a fundamental solution $\tilde{E}$ is constructed s.t. supp $\tilde{E}$ is bounded w.r.t. a part of the variables. The conditions used in the construction cannot be weakened in general. The construction is such that the growth of $\tilde{E}$ is minimal. Regularity and parameter dependence is studied. As an application several existence and regularity results are proved for the inhomogeneous equation.

1. INTRODUCTION

Jones constructed in [7] an elementary solution $\tilde{E}$ for the heat equation such that the support of $\tilde{E}$ is bounded with respect to the time variable. This was used to solve the equation $(\partial/\partial t - \Delta_n)f = g$ such that $\operatorname{supp} f \subset \operatorname{supp} g + \{(x, t) \mid t \in [O, \varepsilon]\}$. In this note we will impose some general conditions on the operator $P(D)$ that imply the existence of an elementary solution whose support is bounded w.r.t. a part of the variables (denoted by x'). This is proved by solving a certain Cauchy problem which "cuts off" a special elementary solution for $P(D)$. In special cases these conditions turn out to be necessary and sufficient for the existence of a fundamental solution with partially bounded support. Especially, for semielliptic operators P there is an elementary solution the support of which is bounded w.r.t. all coordinates x_i s.t. $\deg_{x_i} P < \deg P$, where $\deg_{x_i} P$ is the degree of $P(x)$ in x_i.

The construction naturally violates the uniqueness of the Cauchy problem and so $\tilde{E}$ must have a certain exponential growth at least. This growth is proved to be minimal for a wide class of operators (containing the semielliptic ones) in the sense of the accurate study of Taecklind [13] for the heat equation. The constructed elementary solution is proper if $P \sim \tilde{P}$ and depends holomorphically on the coefficients of P. As an application the support of the solution f of the equation $P(D)f = g$ can be prescribed in terms of $\operatorname{supp} g$, namely, we can get $\operatorname{supp} f \subset \operatorname{supp} g + \{x \mid \|x'\| \leqslant \varepsilon\}$, if $\operatorname{supp} \tilde{E}$ is bounded w.r.t. x'. Several applications are given; especially, this implies that the zero solutions of ρ-hypoelliptic operators on O are of class γ^ρ if $O = O' \times \mathbb{R}^{n-n'}$, where O' is open in $\mathbb{R}^{n'}$.

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2. The Main Construction

In this note the letter P always denotes a polynomial with constant (complex) coefficients in n variables. As usual $P(D)$ is the respective differential operator, where $D := (D_1, \ldots, D_n)$. $D^a := D_1^{a_1} \cdots D_n^{a_n}$, and $|a| := \sum_{i=1}^n a_i$ for $a \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the nonnegative integers. It will be useful for our purposes to divide the coordinates of $y \in \mathbb{R}^n$ in the following way: $y = (x, t) = (x', x'', t)$, where $x = (x', x'') \in \mathbb{R}^{n-1}$, $x' \in \mathbb{R}^{n'}$, $x'' \in \mathbb{R}^{n-n'-1}$, and $t \in \mathbb{R}$. D is handled in the same way. $\mathbb{R}^k$ is always endowed with the supremum norm and B_ε (and B'_ε) denotes the sphere of radius ε centered at 0 with respect to this norm in $\mathbb{R}^n$ (and $\mathbb{R}^{n'}$, respectively). We suppose that P is partially hypoelliptic w.r.t. $t = 0$, that is,

$$P(x, t) := \sum_{k=0}^{d} Q_k(x)\, t^k \qquad \text{and} \qquad Q_d(x) \equiv 1, \tag{2.1}$$

and that P satisfies the following estimates on $\mathbb{R}^n$:

$$|P^{(a)}(x, t)| \leqslant C(1 + |P(x, t)|)\, H(x)^{-|a|/m} \tag{2.2}$$

for some $0 < m < 1$, $C > 0$, and every $a \in \mathbb{N}^{n'}$, where

$$H(x) := 1 + \sum_{j=1}^{d} |Q_{d-j}(x)|^{1/j}.$$

$$H(x)^\gamma \leqslant C(1 + |P(x, t)|) \qquad \text{for some} \quad \gamma > 0 \quad \text{and} \quad C > 0. \tag{2.3}$$

Note that (2.2) implies (2.3) if P is partially hypoelliptic w.r.t. $x' = 0$ (see Theorem 4.2.3III in [4]). The aim of this section is to construct an elementary solution E for $P(D)$ such that the following can be proved: For every $\varepsilon > 0$ there is an elementary solution $\tilde{E}$ for $P(D)$ such that

$$\tilde{E} = E \quad \text{on} \quad B'_\varepsilon \times \mathbb{R}^{n-n'} \qquad \text{and} \qquad \operatorname{supp} \tilde{E} \subset B'_{3\varepsilon} \times \mathbb{R}^{n-n'}. \tag{2.4}$$

Besides the solution of Jones for the heat equation and besides the Schrödinger operator, which can be treated similarly, no further results on this problem seem to be known.

The proof of (2.4) will be done as follows: We will solve the Cauchy problem below on $\mathbb{R}^n$ for special $g \in D(B'_{4\varepsilon} \backslash B'_\varepsilon)$ to cut off the elementary solution E:

$$P(D)\, F = 0 \qquad \text{and} \qquad D_t^i F(x, 0) = D_t^i E(x, 0)\, g(x') \qquad \text{for} \quad 0 \leqslant i \leqslant d-1. \tag{2.5}$$

Note that $D_t^i E(x, 0)$ is defined for $x \neq 0$ by (2.1). If $g \equiv 1$ on $U_\varepsilon := B'_{3\varepsilon} \backslash B'_{2\varepsilon}$, we will show that $F \equiv E$ on $U_\varepsilon \times \mathbb{R}^{n-n'}$. So

$$\begin{aligned} \tilde{E} &:= E - F, \qquad &|x'| < 2,5\varepsilon, \\ &:= 0 &\text{otherwise.} \end{aligned}$$

will be a solution of (2.4).

The main step in solving (2.5) will be the estimation of certain derivatives of $D_t^i E(x, 0)\, g(x')$ with respect to special Sobolev norms which we are going to define. Equation (2.2) implies that $dm \geqslant \deg_{x'} P$ and thus $d > 1$ (the trivial case that P does not depend on x' will be excluded in the sequel). Let $P_-(x, t) := P(-x, -t)$. Then $(1 + |P(x, t)|)^{-k}$ is integrable w.r.t. t for $k \geqslant 1$, as

$$|z|^d \leqslant 2\,|P(x, z)| \qquad \text{for} \quad z \in \mathbb{C} \text{ with } |z| \geqslant 2dH(x). \tag{2.6}$$

Let $\mu_k(x) := \int (1 + |P_-(x, t)|)^{-k}\, dt$ and $\|f\|_{p, \mu_k} := (\int |\hat{f}(y)\, \mu_k(y)|^p\, dy)^{1/p}$. Let $(M_n)_{n \in \mathbb{N}}$ denote an increasing sequence of positive numbers such that

$$M_n \geqslant A^n n! \qquad \text{and} \qquad M_{n'} M_{n''} \leqslant A^{n'+n''} M_{n'+n''}, \tag{2.7}$$

for some $A > 0$ and every $n, n', n'' \in \mathbb{N}$. Let $C^\infty(M_n, \mathbb{R}^k) := \{f : \mathbb{R}^k \to \mathbb{C} \mid |D^a f(x)| \leqslant C_B B^{|a|} M_{|a|}$ for all $B > 0$ and $a \in \mathbb{N}^k\}$. γ^ρ is used instead of $C^\infty(n^{\rho n}, \mathbb{R}^k)$. Using the notation introduced so far we can now state the main result in the proof of (2.4).

THEOREM 2.1. *Let P satisfy* (2.2) *and* (2.3). *Then there is an elementary solution E for $P(D)$ such that for $g \in C^\infty(M_n, \mathbb{R}^{n'})$ and $k \geqslant 1$ there is $k'(k)$ such that for all $h \in D(\mathbb{R}^{n-1})$*

$$|\langle Q(D)\,(D_t^i E(x, 0)\, g), h\rangle| \leqslant C' C^l M_{\{ml'\}+k'} \|h\|_{1, \mu_k},$$

where

$$Q(D) := \prod_{j=1}^{d} Q_{d-j}(D)^{s_j}, \qquad 0 \leqslant l - d - 1 \leqslant \sum_{j=1}^{d} s_j j \leqslant l + d - 1 := l',$$

and

$$\{t\} := \min\{i \in 2\mathbb{Z} \mid i \geqslant t\}.$$

Proof. (a) To start with the construction of E, we first state the existence of some bounded function $G \in C^\infty(\mathbb{R}^n)$ s.t.

$$\begin{array}{lll} |P_-(x,t)| \geqslant 2 & \text{for} & (x,t) \in \operatorname{supp} G, \\ P_- & \text{is bounded on} & \operatorname{supp}(1-G), \\ \nabla_{x'} G & \text{is bounded on} & \mathbb{R}^n. \end{array} \tag{2.8}$$

This is seen as folows: Combining (2.3) and (2.6) gives

$$(H(x) + |t|)^\gamma \leqslant C_1(1 + |P(x,t)|). \tag{2.9}$$

Now $G_1 \in C^\infty(\mathbb{R}^{n-1})$ is chosen that $G_1(x) \equiv 1$ if $H_-(x) \leqslant (3C_1)^{1/\gamma}$ and $\operatorname{supp} G_1 \subset S := \{x \mid H_-(x) \leqslant (3C_1)^{1/\gamma}\} + B_1'$, and such that G_1 and $\nabla_{x'} G_1$ are bounded. Let $G_2 \in D(\mathbb{R})$ satisfy $G_2(t) \equiv 1$ for $|t| \leqslant (4C_1)^{1/\gamma}$. We define $G := 1 - G_1 G_2$. Then $\nabla_{x'} G$ is bounded and $|P(x,t)| \geqslant 2$ for $(x,t) \in \operatorname{supp} G$ by (2.9). As

$$|P_-(x,t)| \leqslant (d+1)(H_-(x)^d + |t|^d), \tag{2.10}$$

the only thing left to prove is that H_- is bounded on S by some constant C_3. This results from (ii) in Lemma 2.2.

LEMMA 2.2. (2.2) *implies each of the equivalent conditions*

(i) $|Q_{d-j}^{(a)}(x)| \leqslant CH(x)^{j-|a|/m}$, *for each* $a \in \mathbb{N}^{n'}$ *and* $x \in \mathbb{R}^{n-1}$.

(ii) $H(x'+y', x'') \leqslant C(H(x', x'') + |y'|^m)$, *for each* $x \in \mathbb{R}^{n-1}$, $y' \in \mathbb{R}^{n'}$.

Proof. (2.2) $\Rightarrow$ (i) Let $a \in \mathbb{N}^{n'}$ and $t := \lambda H(x)$, $|\lambda| \leqslant 1$.

$$\begin{aligned} |P_a(\lambda)| &:= \left| \sum_{k=0}^{d} Q_k^{(a)}(x)\, H(x)^k \lambda^k \right| = |P^{(a)}(x,t)| \\ &\leqslant C(1 + |P(x,t)|)\, H(x)^{-|a|/m} \leqslant C_1 H(x)^{d-|a|/m}, \end{aligned}$$

by (2.2) and (2.10). The proof is completed by noting the fact that for all polynomials $Q(\lambda) := \sum_{|j| \leqslant d} a_j \lambda^j$:

$$\sup_{|j| \leqslant d} |a_j| \leqslant C(d) \sup_{|\lambda| \leqslant 1} |Q(\lambda)|. \tag{2.11}$$

(i) $\Rightarrow$ (ii) This follows by the Taylor formula for $Q_{d-j}(x'+y', x'')$ and the fact that $j \geqslant |a|/m$ if $Q_{d-j}^{(a)} \not\equiv 0$.

(ii) $\Rightarrow$ (i) is proved like (2.2) $\Rightarrow$ (i) by considering

$$Q(\lambda) := Q_{d-j}(x' + \lambda H(x', x'')^{1/m}, x'') \qquad \text{for} \quad \lambda \in \mathbb{R}^{n'}, \quad |\lambda| \leqslant 1.$$

We now choose $M > 2dC_3 + (3C_1)^{1/\gamma}$, where the constants are taken from the above construction of G. This gives by (2.6)

$$|P_-(x, t+zM)| > (2dC_3)^d/2 := C^* > 0 \qquad \text{for} \quad (x, t) \in \operatorname{supp} G_1 G_2$$
$$\text{and } z \in \mathbb{C} \qquad \text{with} \quad |z| = 1. \tag{2.12}$$

Set $E_1 := (2\pi)^{-n} (G/P_-)\hat{}$ ($\hat{}$ being the Fourier transform) and

$$\langle E_2, \varphi\rangle := -i(2\pi)^{-n-1} \int (1 - G(x, t)) \times \int_{|z|=1} \hat{\varphi}(x, t+zM)/P_-(x, t+zM) \frac{dz}{z}\, dx\, dt,$$

for $\varphi \in D(\mathbb{R}^n)$. It is easily seen that $E := E_1 + E_2$ is an elementary solution for $P(D)$. (2.13)

(b) Now E_2 will be estimated. Let $h_1 \in D([-1, 1])$ and $i \in \mathbb{N}$.

$$\begin{aligned}
&|\langle Q(D)(D_t^i E_2 g), hh_1\rangle| \\
&\quad = |\langle E_2, gQ(-D)\, h(-D_t)^i\, h_1\rangle| \\
&\quad = |(2\pi)^{-n-1} \int (1 - G(x, t))\, \hat{g} *' (Q(-D)\, h)\hat{}\ (x) \\
&\qquad \times \int_{|z|=1} (-t - zM)^i \hat{h}_1(t + zM)/P_-(x, t+zM) \frac{dz}{z}\, dx\, dt| \\
&\quad \leqslant C_1 C_2^i \int_{|z|=1} \iint |\hat{g}(y')| \int |Q_-(x' - y', x'')\, \hat{h}(x' - y', x'') \\
&\qquad \times (1 - G(x, t))|\, dx\, dy'\, |\hat{h}_1(t + zM)|\, dt\, dz \\
&\quad \leqslant C_3 C_2^i \iint |\hat{g}(y')| \int H_-(x' - y'), x'')^{l'}\, \hat{h}(x' - y', x'') \\
&\qquad \times (1 - G(x, t))|\, dx\, dy'\, dt\, \sup_{|z|=1} \int |h_1(\tau) \exp(zM\tau)|\, d\tau.
\end{aligned}$$

Here $*'$ denotes the convolution w.r.t. x'. The first inequality follows from (2.12). Using Lemma 2.2(ii) it is seen that $H_-(x' - y', x'')^{l'} \leqslant C^{l'}(C_4^{l'} + |y'|^{l'm})$ on $\operatorname{supp}(1 - G)$. Now the methods of the proof of that lemma can be used to show that the following condition is equivalent to (2.2):

$$|P(x' + y', x'', t)| \leqslant C(1 + |P(x', x'', t)|)(1 + |y'|^m/H(x))^d. \tag{2.14}$$

This implies

$$\mu_k(x', x'') \leqslant C^k(1 + |y'|^{mkd})\, \mu_k(x' - y', x''). \tag{2.15}$$

As μ_k is bounded from below on $\operatorname{supp}(1 - G)$ and P_- is bounded there we finally get for $k' \geqslant \{mkd\} + 2n'$

$$|\langle Q(D)(D_t^i E_2\, g), hh_1\rangle| \leqslant C_5 C^{l'} C_2^i (C_4^l + B^{\{l'm\}+k'} M_{\{l'm\}+k'}) \,\|h\|_{1,\mu_k} \|h_1\|_1.$$

$Q(D)(E_2(\cdot, t)\, g)$ therefore is an entire function of t with values in $B_{\infty, 1/\mu_k}$ and the desired estimate follows.

(c) The estimation of E_1 is a little more involved.

(i) As $\deg_{x'} Q \leqslant ml'$, the Leibniz formula gives

$$\langle Q(D)(D_t^i E_1(\cdot, 0)\, g), h\rangle = \sum_{|a| \leqslant ml'} (-1)^{|a|} (a!)^{-1} \langle Q^{(a)}(D)\, D_t^i E_1(\cdot, 0), D^a gh\rangle. \tag{2.16}$$

Let $\Delta'_\lambda := \sum_{j=1}^{n'} D_j^\lambda$, $k_\lambda(x') := \sum_{j=1}^{n'} x_j^\lambda$ and $h_1 \in D(\mathbb{R})$. We now apply a formula for partial integration by Treves [14, p. 404] on each term in (2.16) separately with even λ. This yields, as $0 \notin \operatorname{supp} g$,

$$\begin{aligned}
&\langle Q^{(a)}(D)\, D_t^i E_1, D^a ghh_1\rangle \\
&\quad = (2\pi)^{-n} \int (-t)^i\, G(x, t)\, Q_-^{(a)}(x)/P_-(x, t)\, \Delta'_\lambda (D^a gh/k_\lambda)\hat{}\ (x)\, dx\, \hat{h}_1(t)\, dt \\
&\quad = (2\pi)^{-n} \int \Bigg[\int (-t)^i\, G(x, t)\, \Delta'_\lambda (Q_-^{(a)}(x)/P_-(x, t))(D^a g/k_\lambda)\hat{}\ *' \hat{h}(x)\, dx \\
&\qquad - \sum_{b=1}^{\lambda} \sum_{j=1}^{n'} \int (-t)^i\, D_j^{\lambda - b}(Q_-^{(a)}(x)/P_-(x, t)) \\
&\qquad \times D_j G(x, t)(x_j^{b-1} D^a g/k_\lambda)\hat{}\ *' \hat{h}(x)\, dx \Bigg] \hat{h}_1(t)\, dt.
\end{aligned} \tag{2.17}$$

If the term in brackets is integrable w.r.t. t then its Fourier transform is continuous and the integral is just $\langle Q^{(a)}(D)\, D_t^i E_1(x, 0), D^a gh\rangle$.

(ii) The sum in (2.17) can be estimated by means of the Cauchy integral formula. Using the Taylor formula, (2.2), and Lemma 2.2(i) it is easily seen that $Q_-(x' + z, x'')$ and $P_-^{-1}(x' + z, x'', t)$ are bounded on $S := \operatorname{supp} D_j G$, if z is taken from a fixed small neighborhood of 0 in $\mathbb{C}^{n'}$. As $D_j G$

and t are bounded on S we can argue as at the end of (b) (using (2.15)) to estimate the sum by

$$C \sum_{b=1}^{\lambda} (\lambda - b)!\, C_1^{|a|+\lambda} M_{|a|+2n'+|mdk|} \|h\|_{1,u_k}. \tag{2.18}$$

(iii) Passing to the first term of (2.17) we will have to estimate $\Delta'_\lambda(Q_-^{(a)}(x)/P_-(x,t))$ for special λ. We state this as a separate lemma.

LEMMA 2.3. *Let P satisfy (2.2) and choose Q as above.*

(a) $|Q_-^{(a)}(x)| \leqslant a! C_1^{|a|+|s|} H_-(x)^{\delta - |a|/m}$, *for* $a \in \mathbb{N}^{n'}$, $\delta = \sum_{j=1}^{d} s_j j$,

(b) $|D_j^{\beta+\eta}(Q_-^{(a)}(x)/P_-(x,t))|$

$\leqslant a!\ (\beta + \eta)!\ C_2^{|a|+\beta+\eta+|s|}\ |P_-(x,t)|^{-1-\eta/(dm)}$, *for* $\beta = \{ml' - |a|\}$, $j \leqslant n'$, *and any* $\eta \in \mathbb{N}$, *if* $|P_-(x,t)| \geqslant 2$.

Proof. (a) For $|s| = 1$ the assertion is given by Lemma 2.2. Set $|s| = k + 1$ and assume that (a) is proved for $|s| \leqslant k$. Take j' s.t. $s_{j'} \geqslant 1$, and set $s'_j := s_j$ for $j \neq j'$, and $s'_{j'} := s_{j'} - 1$,

$$|Q^{(a)}(x)| \leqslant \sum_{\beta \leqslant a} \binom{a}{\beta} \left| Q_{d-j'}^{(\beta)}(x)\, D^{a-\beta} \prod_{j=1}^{d} Q_{d-j}(x)^{s'_j} \right|$$

$$\leqslant a! C \tilde{C}^{|a|+k} H(x)^{\delta - |a|/m} \sum_{\beta \leqslant a} (\tilde{C}^{|\beta|} \beta!)^{-1},$$

where C is taken from Lemma 2.2. This proves (a) if $\tilde{C} \geqslant \max(n', eC)$.

(b) (i) We first note that (2.2) implies

$$|P^{(a)}(x,t)| \leqslant C(1 + |P(x,t)|)^{1-|a|/(dm)} \qquad \text{for} \quad a \in \mathbb{N}^{n'}. \tag{2.19}$$

In fact, using (2.10) we only have to prove

$$|P^{(a)}(x,t)| \leqslant C(1 + |P(x,t)|)(1 + |t|)^{-|a|/m} \qquad \text{for} \quad a \in \mathbb{N}^{n'}. \tag{2.20}$$

This is evident for $|t| \leqslant 2H(x)$ and follows from Lemma 2.2 and (2.6) for $|t| \geqslant 2H(x)$.

(ii) We set $f(x,t) := \min(H(x)^{-1/m}, |P(x,t)|^{-1/(dm)})$ for $|P(x,t)| \geqslant 2$. Reasoning by induction on $|a|$, (2.2) and (2.19) will give (see [14, p. 405])

$$|D^a P_-^{-1}(x,t)| \leqslant a!(AC'f_-(x,t))^{|a|}\, |P_-^{-1}(x,t)| \qquad \text{for} \quad a \in \mathbb{N}^{n'},$$

where A depends on n' only.

(iii) Note that $\deg_{x'} Q^{(a)} \leqslant m\delta - |a|$ for $a \in \mathbb{N}^{n'}$ by (a).

$$
\begin{aligned}
&|D_j^{\beta+\eta}(Q_-^{(a)}(x)/P_-(x,t))| \\
&\quad = \left| \sum_{\gamma \leqslant m\delta - |a|} \binom{\beta+\eta}{\gamma} D_j^{\gamma} Q_-^{(a)}(x)\, D_j^{\beta+\eta-\gamma}(P_-^{-1}(x,t)) \right| \\
&\quad \leqslant \sum_{\gamma} \binom{\beta+\eta}{\gamma} (a+\gamma e_j)!\, C^{|a|+\gamma+|s|} H_-(x)^{\delta-(|a|+\gamma)/m} \\
&\qquad \times (AC'f_-(x,t))^{|\beta+\eta-\gamma|} (\beta+\eta-\gamma)!\, |P_-^{-1}(x,t)|,
\end{aligned}
$$

where e_j is the jth unit vector. Now

$$
f_-(x,t)^{|\beta+\eta-\gamma|} \leqslant H_-(x)^{-\{ml'-|a|\}/m+\gamma/m} |P_-(x,t)|^{-\eta/(dm)},
$$

as $\gamma \leqslant m\delta - |a| \leqslant \beta$. The proof of the lemma is completed by observing that

$$
\sum_{\gamma \leqslant ml'-|a|} \frac{(a+\gamma e_j)!}{\gamma! a!} \leqslant \sum_{\gamma \leqslant \beta} \frac{(|a|+\gamma)^{\gamma}}{\gamma!} \leqslant \exp(|a|+\gamma).
$$

We now choose $\eta := \{md(k-1+(d-1)/\gamma)\}$, where γ is taken from (2.3). As $k+(d-1)/\gamma \leqslant \eta/(dm)+1$ and $|t|^i \leqslant C\,|P_-(x,t)|^{(d-1)/\gamma}$, the application of Lemma 2.3(b) will give

$$
|t^i \Delta'_{\beta+\eta}(Q_-^{(a)}(x)/P_-(x,t))| \leqslant C(\beta+\eta)!\, C_1^{\beta+\eta+|s|+|a|} |P_-^{-1}(x,t)|^k.
$$

Changing the order of integration, the first term of (2.17) is estimated by

$$
\begin{aligned}
&C_2(\beta+\eta)!\, C_1^{|a|+\beta+\eta+|s|} \int |(D^a g/k_{\beta+\eta})^{\wedge}(y')| \int |\mu_k(x',x'')\, \hat{h}(x'-y',x'')|\, dx\, dy' \\
&\quad \leqslant C_3(\beta+\eta)!\, C_4^{\beta+\eta+|s|+|a|} M_{|a|+\{mdk\}+2n'} \|h\|_{1,\mu_k},
\end{aligned}
$$

where C_4 contains the distance from supp g to 0 and (2.15) is used as above.

(iv) In particular we have shown that the integral w.r.t. t in (2.17) exists. Setting $\lambda := \beta + \eta$ in part (ii) of the proof we get

$$
(\lambda - b)!\, M_{|a|+\{mdk\}+2n'} \leqslant A^{\{ml'\}+k'} M_{\{ml'\}+k'},
$$

and

$$
(\beta+\eta)!\, M_{|a|+\{mdk\}+2n'} \leqslant A^{\{ml'\}+k'} M_{\{ml'\}+k'},
$$

where $k' = 2n' + \{mdk + md(k-1+(d-1)/\gamma)\}$ and (2.7) is used. Finally collecting everything we obtain

$$
|\langle Q(D)(D_t^i E(\cdot,0)\, g), h\rangle| \leqslant C_5 C_6^l M_{\{ml'\}+k'} \|h\|_{1,\mu_k},
$$

and Theorem 2.1 is thus proved.

THEOREM 2.4. *Let P satisfy* (2.2) *and* (2.3) *and let E be chosen as in Theorem* 2.1. *Then for every $\varepsilon > 0$ there is an elementary solution $\tilde{E}$ for $P(D)$ such that*

$$\tilde{E} = E \quad \text{on} \quad B'_\varepsilon \times \mathbb{R}^{n-n'} \qquad \textit{and} \qquad \operatorname{supp} \tilde{E} \subset B'_{3\varepsilon} \times \mathbb{R}^{n-n'}.$$

Proof. Following the general line of proof indicated at the beginning of this section, we have to solve the Cauchy problem

$$P(D)\, F = 0, \qquad D_t^i F(x, 0) = D_t^i E(x, 0)\, g \qquad \text{for} \quad 0 \leqslant t \leqslant d-1, \tag{2.5}$$

where $g \in C^\infty(M_n, \mathbb{R}^{n'})$, $\operatorname{supp} g \subset B'_{4\varepsilon} \backslash B'_\varepsilon$, and $g = 1$ on $U_\varepsilon := B'_{3\varepsilon} \backslash B'_{2\varepsilon}$.

(a) To this end it is sufficient to solve the problem

$$P(D)\, L(f) = 0, \qquad D_t^{d-1} L(f)(x, 0) = f(x), \qquad D_t^k L(f)(x, 0) = 0, \tag{2.21}$$

for $0 \leqslant k < d-1$ and special f. In fact,

$$F := \sum_{j=0}^{d-1} \left(\sum_{k=0}^{d-j-1} D_t^k Q_{j+k+1}(D)\, L(D_t^j E(\cdot, 0)\, g) \right), \tag{2.22}$$

will solve (2.5). (2.21) is solved by a power series expansion in t. This yields the following result: Let $\tilde{Q}_{d-1} \equiv 1$ and $\tilde{Q}_l(D) := \sum_s (-1)^{|s|} \prod_{j=1}^d Q_{d-j}(D)^{s_j}$, where the sum is extended over all multi-indices $s := (s_1, \ldots, s_d) \in \mathbb{N}^d$ such that $\sum_{j=1}^d s_j j = l - d + 1$. For $f \in D'(\mathbb{R}^{n-1})$ we define $L(f)(x, t) := \sum_{l=d-1}^\infty \tilde{Q}_l(D) f(x) (it)^l / l!$. If the series converges in $D'_\sigma(\mathbb{R}^{n-1})$ uniformly for bounded t, then it will be a distribution on $\mathbb{R}^n$ that solves (2.5). Note that the coefficient of $(it)^l / l!$ in (2.22) is the sum of at most dl^d terms of the kind considered in Theorem 2.1.

(b) We now choose $M_n := n!^\delta$, where $1 < \delta < 1/m$. γ^δ is a nonquasianalytic class of functions. As μ_k is bounded for large k we may apply Theorem 2.1 and Theorem 2.2.9 in [4] to get

$$\|F(\cdot, t)\|_{\infty, 1} \leqslant C_1 \sum_l C_2^l (\{ml'\} + k')!^\delta \, dl^d \, |t|^l / l! < \infty,$$

as

$$\{ml'\}!^\delta / l! \leqslant C^l l!^{\delta m - 1} \qquad \text{and} \qquad \delta m - 1 < 0.$$

(c) It remains to prove that $F = E$ on $U_\varepsilon \times \mathbb{R}^{n-n'}$. In part (b) of the proof of Theorem 2.1 we already remarked that E_2 is an entire function of t with values in $D'_\sigma(\mathbb{R}^{n'} \backslash \{0\} \times \mathbb{R}^{n-n'})$. This can be proved for E_1 using the method of partial integration in part (c) of that proof and the estimate (2.20). So the equality is established (though we did not assume that $\mathbb{R}^{n-1}$ is noncharacteristic w.r.t. P) and the theorem is proved.

To prove Theorem 2.4 an estimation by Sobolev norms weaker than μ_k clearly would be sufficient. The accurate estimation of Theorem 2.1 is needed for the study of the regularity of $\tilde{E}$ in Section 3. We finish this section by shortly discussing condition (2.2). The first claim is that the constant m in (2.2) can be chosen minimal in $\mathbb{Q}$. In fact, there is a constant $C > 0$ such that

$$C^{-1} < d'(x, t) \sum_{\mathbb{N}^{n'} \ni a \neq 0} |P^{(a)}(x, t)/P(x, t)|^{1/|a|} < C,$$

where

$$d'(x, t) := \inf\{|z - x'| \,|\, P(z, x'', t) = 0, z \in \mathbb{C}^{n'}\}$$

[4, Lemma 4.1.1]. So P satisfies (2.2) iff

$$\sum_{a \in \mathbb{N}^{n'}} |P^{(a)}(x, t)| \leqslant C(1 + |P(x, t)|)$$

and

$$H(x) \leqslant C(1 + d'(x, t))^m.$$

The claim is now proved by the Seidenberg–Tarski lemma [4].

If P_1 and P_2 are equivalent that is,

$C^{-1} \leqslant (1 + |P_1(x, t)|)/(1 + |P_2(x, t)|) \leqslant C$ for some $C > 0$, and if P_1 satisfies (2.2), then so does P_2 with the same m. (2.23)

Considering $P'(\lambda) := P_i(x, \lambda, H_j(x))$, $i \neq j$, as in the proof of Lemma 2.2, it is seen that H_1 and H_2 are equivalent (with obvious notations) and the claim follows from (2.14).

Using (2.23), m can be evaluated easily if P is equivalent to $|P(x', 0, t)| + |P(0, x'', 0)|$ and $P(x', 0, t)$ is semielliptic. In fact, $m = \deg_{x'} P/\deg P$ is the optimal choice in this case. Hence for semielliptic operators there exists a fundamental solution whose support is bounded w.r.t. all variables x_i such that $\deg_{x_i} < \deg_t P$. For x_1-parabolic operators (in the sense of [4]) satisfying (2.2), an elementary solution $\tilde{E}$ can be constructed such that $\operatorname{supp} \tilde{E} \subset [0, C] \times \mathbb{R}^{n-1}$, $C > 0$, as Theorem 2.1 can be proved for $E := (1/P(x_1 + i\tau, x'', t))^\wedge$. This holds especially for operators parabolic in the sense of Petrovski.

No regularity is implied by (2.2) on the x'' variables of the solutions of $P(D)$ whereas P is partially hypoelliptic w.r.t. $x' = 0$ if (2.2) holds, $|P(x, t)| \to \infty$ if $|x'| \to \infty$, and (x'', t) is bounded.

As H is equivalent to the maximum modulus of the complex roots t of

$P_x(t) := P(x, t)$, H_{12} (defined for $P_1 P_2$) is equivalent to $\max(H_1, H_2)$. This can be used to show that in general (2.2) is not preserved by multiplication and therefore is not equivalent to the existence of an elementary solution with x'-bounded support (see Corollary 4.4a)). Consider $P_1(x_1', x_2', t) := x_1'^2 + x_2'^a + t^{a+2}$ and $P_2(x_1', x_2', t) := P_1(x_2', x_1', t)$ where $a > 2$ is even. Equation (2.2) is satisfied by P_i for $m_i = a/(a+2)$ and by $P_1 P_2$ for $m = a^2/(2a+4)$ which is optimal (consider $D_2^a(P_1 P_2)(x_1', 0, 0)$) and exceeds 1 for $a \geqslant 6$. Nevertheless the condition cannot be weakened.

THEOREM 2.5. *Let P be hypoelliptic and equivalent to d_P^d, where $d_P(x, t) := \inf\{|(x, t) - y| \mid P(y) = 0, y \in \mathbb{C}^n\}$ and $d := \deg P$. If all solutions of $P(D)$ are real analytic in t, then the following statements are equivalent:*

(i) *For any $\varepsilon > 0$ there is an elementary solution $\tilde{E}$ of $P(D)$ such that $\operatorname{supp} \tilde{E} \subset B_\varepsilon' \times \mathbb{R}^{n-n'}$.*

(ii) *P satisfies* (1.7).

(iii) *Let O' be open in $\mathbb{R}^{n'}$, $O := O' \times \mathbb{R}^{n-n'}$, and $m < 1$. For $f \in C_P^\infty(O)$ and $(g_j) \in \prod_{j=0}^{d-1} \gamma^{1/m}(O')$ there exists a solution $f \square g \in C_P^\infty(O)$ of the problem $D_t^j(f \square g)(x, 0) = D_t^j f(x, 0)\, g_j(x')$, for $0 \leqslant j \leqslant d-1$ and $x \in O \cap \mathbb{R}^{n-1}$ ("$C_P^\infty(O)$ is a (continuous) $\gamma^{1/m}(O')$ module").*

Proof. (a) The assumptions imply that P is equivalent to $|Q_0(x)| + |t|^d$ (see [10]) and $H(x)^d + |t|^d$. This shows that (2.2) in this case is equivalent to the condition

$$|P^{(a)}(x, t)| \leqslant C(1 + |P(x, t)|)(1 + |t|)^{-|a|/m}, \qquad a \in \mathbb{N}^{n'}. \tag{2.20}$$

Moreover, (2.2) implies that $|Q_{d-j}^{(a)}(x)| \leqslant C(1 + d_P(x, t))^{j - |a|/m}$, which (by a direct generalization of the arguments in Sect. 4 of [4]) gives:

For $K \Subset O$ there exists $C > 0$ such that

$$p(f) := \sup_{(x,t) \in K} \sup_{n_j^a} \prod_{a \in \mathbb{N}^{n'}} \prod_{j=1}^{d} |Q_{d-j}^{(a)}(D)^{n_j^a} f(x, t)| \times C^{-\sum n_j^a} \Big/ \prod_{a,j} n_j^a!^{\,j - |a|/m} < \infty, \tag{2.24}$$

for every $f \in C_P^\infty(O)$.

$$\text{If } O = \mathbb{R}^n, \text{ any } C > 0 \text{ can be chosen in (2.24).} \tag{2.25}$$

Similarly,

$$p_0(f) := \sup_K \sup_a |D_t^a f(x, t)|/(C^a a!) < \infty. \tag{2.26}$$

(b) (iii) $\Rightarrow$ (i) As every elementary solution E belongs to $C_P^\infty(\mathbb{R}^{n'}\setminus\{0\} \times \mathbb{R}^{n-n'})$ and $m < 1$, (iii) implies that E can be "cut off."

(c) (i) $\Rightarrow$ (ii) As will be shown in Section 4, every $f \in C_P^\infty(O)$ coincides with some $F \in C_P^\infty(\mathbb{R}^n)$ on $K' \times \mathbb{R}^{n-n'}$, where $K' \Subset O'$ is arbitrary. (2.26) therefore is valid for f with any $C > 0$. The open mapping theorem shows that p_0 defines a continuous semi-norm on $C_P^\infty(O)$. Applied to $f(x, t) := \exp(i\langle(x_0, t_0), (x, t)\rangle)$, where $P(x_0, t_0) = 0$, $x_0' \in \mathbb{C}^{n'}$, and $(x_0'', t_0) \in \mathbb{R}^{n-n'}$, this gives for any $\varepsilon > 0$

$$\exp(|t_0|) \leqslant \sup_a |t_0|^a 2^a/a! = p_0(f) \leqslant C(\varepsilon) \sup_{\substack{|x'| < \varepsilon \\ |(x'', t)| \leqslant B(\varepsilon)}} |f(x, t)|$$

$$\leqslant C_1(\varepsilon) \exp(\varepsilon |\mathrm{Im}\, x_0'|). \tag{2.27}$$

Now define $\mu(\tau) := \inf\{|\mathrm{Im}\, x'| \mid P(x', x'', t) = 0,\ t^2 - \tau^2 \leqslant 0\}$. By (2.27) we have $\tau \leqslant \varepsilon\mu(\tau)$ for any $\varepsilon > 0$ and large τ, whereas $\mu(\tau) = A\tau^a(1 + o(1))$, $A > 0$ and $a \in \mathbb{R}$, by the Seidenberg–Tarski lemma. This shows that a is strictly larger than 1 and $|t| \leqslant C(1 + |\mathrm{Im}\, x'|)^m$ for some $m < 1$ if $P(x', x'', t) = 0$. Now $|t| \leqslant C(1 + d'(x', x'', t))^m$ for $(x, t) \in \mathbb{R}^n$ and this shows that (2.20) and therefore (2.2) is valid.

(d) (ii) $\Rightarrow$ (iii) Let us suppose first that $f \in C_P^\infty(\mathbb{R}^n)$ and $g_j \in \gamma^{1/m}(\mathbb{R}^{n'})$. We have shown in Theorem 2.4 that we need an estimation for $Q(D)(D_t^i f(x, 0)\, g_i(x'))$ where $Q(x) := \prod_{j=1}^d Q_{d-j}(x)^{s_j}$ and $\sum_{j=1}^d s_j j \leqslant l' = l + d - 1$. The following product formula is proved by indicution (see Sect. 4 in [11] for the case $d = 1$):

$$|Q(D)\,(D_t^i f(x, 0)\, g_i(x'))| \leqslant r^{|s|} \sup \prod_{j=1}^{d} \prod_{a \in A_j} |Q_{d-j}^{(a)}(D)^{n_j^a}\, D_t^i f(x, 0)\, D^\beta g(x')|,$$

where $A_j := \{a \in \mathbb{N}^{n'} \mid Q_{d-j}^{(a)} \not\equiv 0\}$, $r := \sum_j |A_j|$, $\beta := \sum_{a \in A_j} n_j^a a$, and the supremum is taken over all n_j^a s.t. $\sum_{a \in A_j} n_j^a = s_j$. The term with respect to f can be estimated uniformly on every compact set by $C(C_1)(C_1 l)^{l' - \sum n_j^a |a|/m}$ for any $C_1 > 0$ (using (2.24) and (2.25)). As $g \in \gamma^{1/m}$ the power series converges for any t. The theorem is thus proved in this case. As (ii) implies (i) the general case can be handled as in (c) such that the above argumentation is applicable.

Note that Theorem 2.5 is a first application of the results of this section. It should be remarked that it is now easy to show that it is not sufficient for the existence of an elementary solution with x-bounded support that $\deg_x P < \deg_t P$, though (2.5) can be solved in certain ultradistribution spaces.

3. Growth, Regularity, and Parameter Dependence

The construction of an elementary solution, the support of which is bounded with respect to the x' variables, hurts the uniqueness of the Cauchy problem with data for $x_i = 0$ $(1 \leqslant i \leqslant n')$. So $\tilde{E}$ clearly has to satisfy a certain exponential growth at least. The study of the growth of $\tilde{E}$ will be carried out with respect to certain Sobolev norms in general. For hypoelliptic operators an estimation with respect to the supremum norm is implied which generalizes the estimations proved in [7] for the heat equation to the class of operators considered in Section 2. We keep all notations of this section.

To get a better insight into the different growth of $M_{[ml]+k}$ (in Theorem 2.1) and $l!$ and to minimize the frowth of M_n we use the following construction [7, Appendix; 1, p. 269]:

For $m < 1$ let h be a positive and monotonically increasing function defined on $[0, \infty)$, such that

$$\int_0^\infty h(t)^{1-1/m}\, dt < \infty. \tag{3.1}$$

Let f be the inverse function of $th(t)$. If $y = sh(s)$ then $f(y)/y = s/(sh(s)) = 1/h(s)$. So $f(y)/y$ is decreasing monotonically. Equation (3.1) is equivalent to the existence of the series $\sum_{k=1}^\infty (f(k)/k)^{1/m} := C^*$. (See [1, p. 269]; the assumption "$h(t) \geqslant Ct^{m/(1-m)}$" used there is redundant as this estimation is needed for some sequence $t_n \to \infty$ only. But this follows from (3.1).)

There exists a sequence $(b_k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} b_k = \infty$, $(b_k f(k)/k)^{1/m}$ is decreasing monotonically, and $\sum_{k=1}^\infty (b_k f(k)/k)^{1/m} < \infty$. Choose for instance b_k maximal such that $b_k \leqslant 2^{nm}$ if $C^* 4^{-n-1} \leqslant \sum_{j \geqslant k} (f(j)/j)^{1/m} \leqslant C^* 4^{-n}$ and such that $(b_k f(k)/k)^{1/m}$ decreases.

M_n is defined inductively: $M_0 = 1$, $M_{n-1}/M_n := (b_n f(n)/n)^{1/m}$ for $n \geqslant 1$. M_n is logarithmically convex and $M_{n-j} M_j \leqslant M_n$ for $j \leqslant n$. As by assumption $\sum_{n=1}^\infty M_{n-1}/M_n$ exists and M_{n-1}/M_n decreases, nM_{n-1}/M_n is bounded and $M_n \geqslant C^n n!$ for some $C > 0$. Moreover the class of functions $C^\infty(M_n, \mathbb{R}^{n'})$ is nonquasianalytic. Summarizing, the sequence M_n satisfies all conditions needed in the construction of $\tilde{E}$ in Section 2.

Theorem 3.1. *Let P satisfy* (2.2) *and* (2.3) *and h be chosen as above. Then the elementary solution $\tilde{E}$ constructed for $P(D)$ in Section* 2 *satisfies the following estimation: For all $C > 0$ there exists $a > 0$ such that*

$$\|E(\cdot, t)\|_{\infty, 1/v} \leqslant a \exp(C\,|t|\, h(|t|)), \tag{3.2}$$

where

$$v(x) := \int (1 + |P(x, t)|)^{-1}\, dt.$$

Proof. (i) First note that

$$M_n \leqslant (n/(b_n f(n)))^{n/m}$$

and

$$M_{\{ml'\}+k'} \leqslant b_{l+\gamma}^{-l-\gamma}((l+\gamma)/f(l+\gamma))^{l+\gamma} \qquad \text{for some} \quad \gamma \geqslant k'/m, \tag{3.3}$$

as $k/(b_k f(k))$ increases. We now consider the solution F of the Cauchy problem (2.5). The proof is similar to that in [7]. Using (3.3) and Theorem 2.1 we get for $k \geqslant 1$, $|t| \geqslant 1$, and $v_k := (\mu_k)_-$

$$\|F(\cdot, t)\|_{\infty, 1/v_k} \leqslant C \sum_{l=0}^{\infty} ((l+\gamma)/f(l+\gamma))^{l+\gamma} (l+\gamma)!^{-1} \times (C' |t|/b_{l+\gamma})^{l+\gamma}. \tag{3.4}$$

Set $y = |t|\, h(|t|)$, that is, $f(y) = |t|$ and $y/f(y) = h(|t|)$.

(a) As $\lim_{b\to\infty} b_k = \infty$, (3.4) can be estimated by

$$\sum_{l=0}^{\infty} (C' |t|\, h(|t|)/b_{l+\gamma})^{l+\gamma} (l+\gamma)!^{-1} \leqslant a(C) \exp(C\, |t|\, h(|t|)),$$

for any $C > 0$, if $l + \gamma \leqslant y$.

(b) For $l + \gamma \geqslant y$, (3.4) is bounded by $C \sum_{l=0}^{\infty} (eC'/b_{l+\gamma})^{l+\gamma}$, as f increases and $\lim_{k\to\infty} b_k = \infty$.

(c) For $|t| \leqslant 1$, $\|F(\cdot, t)\|_{\infty, 1/v_k}$ can be estimated as in (b). Summarizing, we get for any $t \in \mathbb{R}$ and $C > 0$

$$\|F(\cdot, t)\|_{\infty, 1/v_k} \leqslant a(C) \exp(C\, |t|\, h(|t|)). \tag{3.5}$$

(ii) (a) As $(1 + |P_-(x, t)|)^{-1}$ is integrable with respect to t it is easy to show that $|\langle E_1(\cdot, t), \varphi\rangle \leqslant C\, \|\varphi\|_{1,\mu_1}$ for every $\varphi \in D(\mathbb{R}^{n-1})$, that is, $\|E_1(\cdot, t)\|_{\infty, 1/v} \leqslant C < \infty$ for any t.

(b) Using similar methods as in part (b) of the proof of Theorem 2.1 we get for $\varphi \in D(\mathbb{R}^{n-1})$ and $\psi \in D(\mathbb{R})$

$$|\langle E_2, \varphi\psi\rangle| \leqslant C\, \|\varphi\|_{1,\mu} \int |\psi(t) \exp((1+t^2)^{1/2} M)|\, dt,$$

that is,

$$\|E_2(\cdot, t)\|_{\infty, 1/v} \leqslant C \exp(M(1+t^2)^{1/2}),$$

for any t.

(iii) The theorem follows from (i) and (ii) and the fact that h is unbounded.

If P satisfies the further condition that $H(x) \to \infty$ for $|x| \to \infty$, then by the Sobolev lemma F is an infinitely differentiable function and the estimation (3.5) holds with respect to the supremum norm, as we could take any k in part (i) of the proof above. This especially holds for hypoelleptic P where the further estimation $|E(x, t)| \leqslant C \exp(M(1+t^2)^{1/2})$, for $|(x, t)| \geqslant \delta > 0$, can be achieved in this case. We therefore have proved

COROLLARY 3.2. *If P is hypoelliptic and satisfies* (2.2), *then*

$$\sup_{|(x,t)| \geqslant \delta} |\tilde{E}(x, t) \exp(-C\,|t|\,h(|t|))| < \infty \qquad \textit{for any} \quad \delta, C > 0. \qquad (3.6)$$

This is the estimate proved in [7] for the heat equation (where in [7] only the existence of some $C > 0$ is shown). Namely, for this equation we have $m = \frac{1}{2}$ and thus $1 - 1/m = -1$.

We will show now that the growth of $\tilde{E}$ proved above is minimal for a wide class of operators. Let $\tilde{E}$ and $\tilde{E}'$ be two elementary solutions constructed as in Section 2, the support of which is contained in $B'_{3\varepsilon} \times \mathbb{R}^{n-n}$ and $B'_{3\varepsilon'} \times \mathbb{R}^{n-n'}$, respectively. The difference $\tilde{E} - \tilde{E}'$ is a zero solution of $P(D)$ which satisfies the growth condition (3.6) on $\mathbb{R}^n$. The support of $\tilde{E} - \tilde{E}'$ is bounded with respect to x'. Now the following statement is shown in [12]: Let P be hypoelliptic and $|D_t^a P(x, t)| \leqslant C(1 + |P(x, t)|)(1 + |x_j|)^{-a/r}$ for any $(x, t) \in \mathbb{R}^n$ and $a \in \mathbb{N}$. Then every $f \in C_P^\infty(\mathbb{R}^n)$ vanishes identically provided that $|f(x, t)| \leqslant a \exp(C\,|t|\,h(|t|))$ on $\mathbb{R}^n$ for some C, $a > 0$, where $\int h(t)^{1-r}\,dt = \infty$, and that $\operatorname{supp} f$ is bounded with respect to x_j. Now if P is equivalent to $|P(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n-1}, 0)| + |x_j|^{d'} + |t|^d$ for some $j \leqslant n'$, then the considerations of Section 2 show that $1/r = d'/d = m$ ($d' = \deg_{x'} P$). Consequently, an estimation like (3.6) can be proved for $\tilde{E}$ in this case iff $\int h(t)^{1-1/m}\,dt$ exists.

We now turn to the study of the regularity of E and $\tilde{E}$, respectively. Let $\|\;\|_P := \|\;\|_{1,1/(1+|P|)}$ and $\varphi \in D(\mathbb{R}^n)$. If $\psi \in D(\mathbb{R})$ is chosen such that $\varphi\psi = \varphi$, then the following is evident:

$$|\langle E_1, \varphi\rangle| \leqslant C\,\|\varphi\|_P$$

and

$$|\langle E_2, \varphi\rangle| \leqslant C_1\,\|\varphi\|_P \sup_{|z|=1} \|\exp(tzM)\,\psi\|_1 \leqslant C_2\,\|\varphi\|_P\,.$$

Thus E fulfills: For every $a > 0$ there exists $C > 0$ such that

$$|\langle E, \varphi\rangle| \leqslant C\,\|\varphi\|_P \qquad \text{for all} \quad \varphi \in D(\mathbb{R}^{n-1} \times (-a, a)). \qquad (3.7)$$

The same estimate can be proved for F as follows: Let $\hat{}^t$ be the Fourier transform with respect to t. Let $\varphi \in D(\mathbb{R}^{n-1} \times (-a, a))$, $\psi \in D(]-a_1, a_1[)$, $\psi \equiv 1$ on $[-a, a]$, and $i \geqslant d/2$. As $D_t^l F$ can be estimated like F itself we get

$$\begin{aligned}|\langle F, \varphi\rangle| &= |\langle (1 + D_t^{2i})(\psi F), (\hat{\varphi}^t/(1 + t^{2i}))\check{}_{-}^{t}\rangle|\, 2\pi \\ &\leqslant C \exp(a_1 h(a_1)) \sup_{|\tau| \leqslant a_1} \|(\hat{\varphi}^t/(1 + t^{2i}))^{\wedge t} (\cdot, -\tau)\|_{1, \mu_\eta}. \qquad (3.8)\end{aligned}$$

The inequality

$$\begin{aligned}|(\hat{\varphi}^t/(1 + t^{2i}))^{\wedge t \wedge x}(x, \tau)| &= |(\hat{\varphi}/(1 + t^{2i}))^{\wedge t} (x, \tau)| \\ &\leqslant \int |\hat{\varphi}(x, t)/(1 + t^{2i})|\, dt,\end{aligned}$$

shows that

$$\sup_{|\tau| \leqslant a_1} \|(\hat{\varphi}^t/(1 + t^{2i}))^{\wedge t} (\cdot, \tau)\|_{1, \mu_\eta} \leqslant \|\varphi\|_{1, v_\eta}, \qquad v_\eta(x, t) := \mu_\eta(x)/(1 + t^{2i}).$$

Equations (2.9) and (2.10) imply that for sufficiently large η, $v_\eta(x, t)$ is smaller than $C(1 + |P_-(x, t)|)^{-1}$, that is, $\|\varphi\|_{1, v_\eta} \leqslant \|\varphi\|_{P_-}$.

Equation (3.7) is thus valid for F and $E - F$. (3.9)

If P is equivalent to $\tilde{P}$ we can conclude that $\tilde{E}$ belongs to $B^{\mathrm{loc}}_{\infty, \tilde{P}}$, that is, $\tilde{E}$ has optimal regularity (3.10)

[4, p. 70]. If $P \not\sim \tilde{P}$, then $B_{\infty, (1 + |P_-|)}$ is not semilocal in the sense of [4]. The respective local space thus cannot be defined. As a substitute we can prove the following theorem (see [14, Theorem 3.2]):

THEOREM 3.3. *Let* $L_2^c := \{f \in L_2 \mid \operatorname{supp} f \Subset \mathbb{R}^n\}$ *and* $L_2^{c/\mathrm{loc}} := \{f \in L_2^{\mathrm{loc}} \mid \operatorname{supp} f$ *is bounded with respect to* $x'\}$ *be endowed with their natural inductive limit topologies. Then* $Q(D) \tilde{E} *$ *is a continuous linear operator from* L_2^c *into* $L_2^{c/\mathrm{loc}}$ *for any* Q *such that*

$$|Q(y)| \leqslant C(1 + |P(y)|) \qquad on \quad \mathbb{R}^n. \qquad (3.11)$$

Proof. (a) Q is called weaker than P ("$Q \lesssim P$" if it satisfies (3.11). This order relation is finer than the relation "$\tilde{Q} \lesssim \tilde{P}$" considered in [4]. Now (3.9) implies for $v, w \in D(\mathbb{R}^n)$

$$
\begin{aligned}
|\langle Q(D)(E-F) * v, w\rangle| &= |\langle E-F, (Q(D)\,v)\check{\ } * w\rangle| \\
&\leqslant C \int |Q_-(y)/(1+|P_-(y)|)\,\hat{v}_-(y)\,\hat{w}(y)|\,dy \\
&\leqslant C_1 C\,\|v\|_2\,\|w\|_2, \qquad (3.12)
\end{aligned}
$$

where C depends on the supports of v and w only.

(b) As L_2^c is a strict inductive limit of Hilbert spaces it is complete, regular, and reflexive [15]. L_2^{loc} is a (F)-space and $(L_2^c)'_b = L_2^{\text{loc}}$ as is easily seen. Every bounded subset of L_2^c and L_2^{loc} is contained in the closure of a subset of $D(\mathbb{R}^n)$ bounded in L_2^c and L_2^{loc}, respectively. Now (3.11) just means that $Q(D)(E-F)*$ is a continuous operator from L_2^c into L_2^{loc}. If $v \in L_2^c$ and $\operatorname{supp} v \subset B'_{\varepsilon/3} \times \mathbb{R}^{n-n'}$, then the supports of $\tilde{E} * v$ and $(E-F-\tilde{E}) * v$ are disjoint. This shows that $Q(D)\,\tilde{E}*$ is continuous from $L_2^c(B'_{\varepsilon/3} \times \mathbb{R}^{n-n'})$ into L_2^{loc}. Using a suitable resolution of the identity and the fact that $\operatorname{supp} \tilde{E}$ is bounded with respect to x', one immediately shows the theorem.

Theorem 3.3 states that $\tilde{E}$ has optimal regularity with respect to the order relation $\lesssim$.

At last we want to study the dependence of the statements proved so far on the coefficients of P, that is, we consider $P(y) = \sum_a z_a y^a$ as a function of its coefficients. To stress this fact we sometimes denote P by P_z.

The set $W(P)$ of all polynomials $Q \lesssim P$ apparently is a vector space and depends on the equivalence class Γ of P only. Q_z is often identified with the coefficient vector (z_a). So $W(P)$ carries the natural norm topology of a finite dimensional vector space.

The polynomials equivalent to P form a circled ($aQ \sim P$ for all $a \in \mathbb{C}$ if $Q \sim P$) open subset of $W(P)$. For $P' \sim P$ consider the norm $\|Q\|^{P'} := \sup_y |Q(y)|/(1+|P'(y)|)$.

$$
\text{If } \|Q-P'\|^{P'} := N < 1, \qquad \text{then} \quad \|P'\|^{Q} < (1-N)^{-1}, \qquad (3.13)
$$

and therefore $Q \lesssim P' \lesssim Q$ (see [14, p. 226]), that is, $Q \sim P$.

We now consider the conditions of Section 2. If P is partially hypoelliptic with respect to $t=0$, then so is every $Q \in \Gamma$ and $\deg_t Q = d = \deg_t P$. The coefficient $c(d,Q)$ of t^d in $Q(x,t)$ never vanishes for $Q \in \Gamma$. We may thus assume that $c(d,Q)=1$ when we are considering the dependence of the constructions of Section 2 on the coefficients of Q.

If P satisfies (2.2) so does every $P' \in \Gamma$ with the same constant m by (2.23). Moreover, if $U := \{Q \mid \|Q-P'\|^{P'} < \frac{1}{2}\}$, then

$$
H_Q(x) \leqslant C' H_{P'}(x),
$$

and

$$
|Q(x'+y', x'', t)| \leqslant C''(1+|Q(x',x'',t)|)(1+|y'|^m/H_Q(x',x''))^d,
$$

for all $(x, t) \in \mathbb{R}^n$, $y' \in \mathbb{C}^{n'}$, and every $Q \in U$. In fact, the constants in Lemmas 2.2 and 2.3 can be chosen uniformly for $Q \in U$ as well as the constant appearing in (2.3). All constants used in the construction of Section 2 thus can be chosen uniformly for $Q \in K \Subset \Gamma$, which implies

THEOREM 3.4. *Let P satisfy* (2.2) *and* (2.3) *and let Γ be the equivalence class of P. For $K \Subset \Gamma$ there exists a function $\tilde{E}_z : K \to D'(\mathbb{R}^n)$, such that $\tilde{E}_z$ is an elementary solution of $P_z(D)$ for every $z \in K$ and*

(a) $\operatorname{supp} \tilde{E}_z \subset B'_{3\varepsilon} \times \mathbb{R}^{n-n'}$,

(b) *the following mappings are holomorphic.*

(i) $z \to \tilde{E}_z$, $\mathring{K} \to D'(\mathbb{R}^n)$,

(ii) $z \to \tilde{E}_z v_c^{-1}$, $\mathring{K} \to S'_b(\mathbb{R}^n)$,

(iii) $z \to Q(D)\tilde{E}_z$, $\mathring{K} \to L_b(L_2^c, L_2^{c/\mathrm{loc}})$, *for* $Q \lesssim P$, *where* $v_c(t) := \exp(c(1+t^2)^{1/2} h_1((1+t^2)^{1/2}))$ *for some C^∞ function $h_1 \leqslant h$ satisfying* (3.1).

Proof. Only (b)(i) is proved. The other statements follow by a similar reasoning (see [14]). As the function G in (2.8) and the constant M in the definition of E_2 can be chosen independent of $Q \in K$, the Cauchy integral theorem and the theorem of Fubini easily show that E_z is weakly holomorphic with values in $D'(\mathbb{R}^n)$. Let F_z be the solution of (2.5) for $P_z(D)$ and E_z with g independent of z. By the above remarks the power series $F_z(\cdot, t)$ converges absolutely and uniformly for $z \in K$ and bounded t. So F_z is a (weakly) holomorphic function of $z \in \mathring{K}$ because the data in (2.5) depend holomorphically on z. E_z, F_z, and $\tilde{E}_z$ are bounded on K. This implies that they are holomorphic on $\mathring{K}$ as $D'_b(\mathbb{R}^n)$ is (sequentially) complete.

The statement of the theorem is semiglobal contrary to the local statements of Theorems 3.4 and 3.5 in [14]. Using a partition of unity one can prove that there is a function $\tilde{E}_z$ defined on Γ, satisfying the theorem above with "holomorphic" replaced by „infinitely differentiable" (see [14, p. 225]). Localized versions hold if z is varying on a holomorphic (C^∞, respectively) manifold.

4. APPLICATIONS

In this section we are always assuming that for every $\varepsilon > 0$ there is an elementary solution $\tilde{E}$ for $P(D)$ s.t. the support of $\tilde{E}$ is contained in

$$K_\varepsilon := B'_\varepsilon \times \mathbb{R}^{n-n'}. \tag{4.1}$$

We are going to solve the equation $P(D)f = g$ such that $\operatorname{supp} f \subset$

$\operatorname{supp} g + K_\varepsilon$. As in the case of parabolic operators a density lemma for special solutions of $P(D)$ is needed.

LEMMA 4.1. *Let $P(D)$ satisfy* (4.1). *Let O_1, O_2 be open and bounded sets in $\mathbb{R}^n$ and $O_1 \subset O_2$. For $j = 1, 2$ and $c > 0$ let $N_j^c := \{f \in C_P^\infty(O_j) \mid \operatorname{supp} f \subset \bar{K}_c\}$ with the topology induced by $C^\infty(O_j)$. Let $b > a$ and suppose that if $T \in D'(\mathbb{R}^n)$, $K_a \cap \operatorname{supp} T \subset \bar{O}_2$, and $K_b \cap \operatorname{supp} P(-D)\, T \Subset O_1$, then*

$$K_a \cap \operatorname{supp} T \Subset O_1. \tag{4.2}$$

Then N_1^a is contained in the closure (in $C^\infty(O_1)$) of the restrictions to O_1 of the functions in N_2^b.

Proof. The proof follows that of Theorem 5.8.3 in [4] and will be sketched only. We have to show that every $v \in \mathscr{E}'(O_1)$ vanishing on the restrictions of N_2^b vanishes on N_1^a. Let $\tilde{E}$ be an elementary solution of $P(D)$ with $\operatorname{supp} \tilde{E} \subset K_{(b-a)/2}$. One easily shows that $\mu := \tilde{E} * v$ vanishes on $K_a \cap O_2$ which implies that $\operatorname{supp} \mu \cap K_a \Subset O_1$ by (4.2). Using a suitable $\varphi \in D(O_1)$ as in the theorem cited above we get

$$v|_{N_1^a} = P(D)(\varphi\mu)|_{N_1^a} = 0.$$

We will show now that Lemma 4.1 is fulfilled for a certain system of convex sets. For this purpose we define $V := \{(v_1, v_2) \mid v_1 \in \mathbb{R}^{n'}, v_2 \in \mathbb{R}^{n-n'},$ (w, v_2) is noncharacteristic with respect to P for all $w \in \mathbb{R}^{n'}\}$ and $O_V := \{O \mid O = \bigcap_{i \in I} H_i$, where H_i are half spaces s.t. ∂H_i is parallel to $\langle v_i \rangle^\perp$, $v_i \in V\}$.

LEMMA 4.2. *Let O_1, $O_2 \in O_V$ be open and bounded and $O_1 \subset O_2$. Then the assumptions of Lemma* 4.1 *are fulfilled.*

Proof. Let T be given as in (4.2) and $y \in CO_1 \cap K_a$. By assumption there exists a hyperplane $H := y_2 + \langle v \rangle^\perp$ where $v \in V$, which separates y and $\operatorname{supp} T \cap K_a$. Define H_2 to be the half space with $\partial H_2 = H$ containing y and set $\Omega_2 = H_2 \cap K_a$. Now choose the half space H_1 with $\partial H_1 = y_1 + \langle v \rangle^\perp$ such that $H_1 \cap O_2 = \varnothing$. For $\Omega_1 := H_1 \cap K_a$ every characteristic hyperplane intersecting Ω_1 intersects Ω_2 by the choice of v. Now T is a zero solution defined on Ω_2 and vanishing on Ω_1. As Ω_1 and Ω_2 fulfill the assumptions of Theorem 5.3.3 in [4], T is vanishing on Ω_2, that is, in a neighborhood of y. This completes the proof.

THEOREM 4.3. *Let P satisfy* (4.1) *and let V contain a basis of $\mathbb{R}^n$.*

(a) *For* $O \in O_V$ *and* $g \in C^\infty(O)$ *the equation* $P(D)f = g$ *is solvable with*

$$f \in C^\infty(O) \qquad and \qquad \operatorname{supp} f \subset \operatorname{supp} g + K_\varepsilon. \tag{4.3}$$

As $\mathbb{R}^n \in O_V$, *we may choose* $O = \mathbb{R}^n$.

(b) *If* V *is dense in* $\mathbb{R}^n$, *then* (a) *is valid for every convex set.* (4.4)

(c) *Let* $k_s(x) := (1 + |x|^2)^{s/2}$. *Suppose that there exists* s_0 *s.t.* (4.1) *is fulfilled with*

$$\tilde{E} \in B^{\mathrm{loc}}_{\infty, k_{s_0}}. \tag{4.1'}$$

Then (a) *and* (b) *are valid for* $g \in D'_F(O)$ *with* $f \in D'_F(O)$.

Proof. (a) Using a suitable resolution of the identity in $\mathbb{R}^{n'}$ it is easy to see that it suffices to solve the above equation for supp $g \subset K_a$ with $\operatorname{supp} f \subset K_b$, $b > a$. As V contains a basis of $\mathbb{R}^n$ there exists a sequence $\Omega_k \in O_V$ of bounded open sets s.t. $\Omega_k \subset \bar{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup \Omega_k = \mathbb{R}^n$. For $O \in O_V$ such a sequence $O_k \in O_V$ apparently also exists. Let $\tilde{O}_k := O_k \cap \Omega_k$. $\tilde{O}_k \in O_V$ is bounded. Let p_k be an increasing system of semi-norms with support in $\tilde{O}_k$ defining the topology of $C^\infty(O)$. As usual a sequence $f_k \in C^\infty(\tilde{O}_k)$ can be defined (using (4.1) and Lemma 4.1) s.t.

(i) $P(D)f_k = g$ on $\tilde{O}_{k-1}$,

(ii) $\operatorname{supp} f_k \subset \tilde{O}_k \cap K_{a_k}$, where $a_k := (b-a)\sum_{j=1}^{k} 2^{-j} + a$,

(iii) $p_k(f_k - f_{k-1}) < 2^{-k}$.

By (iii) f_k converges on O against $f \in C^\infty(O)$. f solves $P(D)f = g$ by (i) and $\operatorname{supp} f \subset K_b$ by (ii).

(b) is obvious as O_V coincides with the system of all convex sets.

(c) Let $H^{\mathrm{loc}}_s := B^{\mathrm{loc}}_{2,k_s}$. As the regularisations of a distribution $T \in H^{\mathrm{loc}}_s$ converge against T in H^{loc}_s and C^∞ has a stronger topology, one can prove a statement similar to Lemma 4.2 and valid in H^{loc}_s. Now $g \in D'_F(O)$ belongs to $H^{\mathrm{loc}}_{s_1}$ for some s_1 and the construction in (a) is going on in $H^{\mathrm{loc}}_{s_0+s_1}$ as $\varphi g * \tilde{E} \in H^{\mathrm{loc}}_{s_0+s_1}$ for $\varphi \in D(\tilde{O}_k)$. This shows the theorem.

If P is hypoelliptic, then Theorem 4.3 is valid for every distribution f.

Equation (4.4) is valid if the principal part of P is independent of x' or if there is $i \in \mathbb{N}$ s.t. $x_i = 0$ for all characteristic vectors x, especially, if P is hypoelliptic and all zero solutions of P are real analytic in x_i.

If (4.1) is valid with $n' = n - 1$ then there is a zero solution with support bounded with respect to x. We may suppose P to be irreducible. By Theorem 5.7.1 in [4] the principal part of P is hyperbolic with respect to (x, t) if $t \neq 0$.

Especially, (x, t) is noncharacteristic for $t \neq 0$ and so (4.4) is fulfilled. This also follows from (2.2). In fact, as a consequence of (2.19) we have $|P(x'+y', x'', t)| \leqslant C(1+|P(x,t)|+|y'|^{md})$. Therefore (x', x'', t) is characteristic iff (y', x'', t) is characteristic for any $y' \in \mathbb{R}^{n'}$. Thus V is the set of all noncharacteristic vectors which is clearly dense in $\mathbb{R}^n$. As (4.1′) was proved in Section 2, (2.2) thus implies that (4.3) can be solved for convex sets O and $g \in D'_F(O)$.

COROLLARY 4.4. (a) *Let V_1 (defined for P_1) contain a basis of $\mathbb{R}^n$. Then* (4.1) *is valid for $P_1 P_2$ if* (4.1′) *is valid for P_1 and P_2.*

(b) *Let V contain a basis of $\mathbb{R}^n$ and P satisfy* (4.1′). *Then for every zero solution $f \in D'_F(O' \times \mathbb{R}^{n-n'})$, O' open in $\mathbb{R}^{n'}$, and every $K' \Subset O'$, there is a solution $F \in D'_F(\mathbb{R}^n)$ s.t.*

$$F|_{K' \times \mathbb{R}^{n-n'}} = f \qquad \textit{and} \qquad \operatorname{supp} F \subset O' \times \mathbb{R}^{n-n'}.$$

(c) *Same assumptions as in* (b). *Then for every elementary solution E there is an elementary solution $\tilde{E}$ s.t. $\tilde{E} = E$ on $B_\varepsilon \times \mathbb{R}^{n-n'}$ and* $\operatorname{supp} \tilde{E} \subset B_{3\varepsilon} \times \mathbb{R}^{n-n'}$.

Proof. (a) If (4.1′) is fulfilled for P_1 and P_2 with $\tilde{E}_i \in B^{\mathrm{loc}}_{\infty, k_{\nu_i}}$, then the equation $P_1(D)\tilde{E} = \tilde{E}_2$ is solvable in D'_F such that $\operatorname{supp} \tilde{E} \subset B_{2\varepsilon} \times \mathbb{R}^{n-n'}$ as Theorem 4.3(c) shows.

(b) One has to choose $h \in C^\infty(\mathbb{R}^{n'})$ suitably and to solve $P(D)g = P(D)(fh)$ using Theorem 4.3. $F := fh - g$ then solves (b).

(c) is a direct application of (b).

The growth condition proved in Section 3 for the "cut off" elementary solution $\tilde{E}$ naturally cannot be expected in Corollary 4.4(c).

If P is ρ-hypoelliptic then a classical theorem of Hoermander states that $C_P^\infty(O) \subset \Gamma^\rho(O)$. We will show now that for certain sets O we have $C_P^\infty(O) \subset \gamma^\rho(O)$ indeed.

COROLLARY 4.5. *Let P be hypoelliptic and satisfy* (4.1). *If O' is open in $\mathbb{R}^{n'}$, then $C_P^\infty(O' \times \mathbb{R}^{n-n'}) \subset \gamma^\rho(O' \times \mathbb{R}^{n-n'})$.*

Proof. By Corollary 4.4(b) we may assume that $f \in C_P^\infty(O' \times \mathbb{R}^{n-n'})$ is defined on $\mathbb{R}^n$ in fact. But as we noticed in Section 2 already, $C_P^\infty(\mathbb{R}^n) \subset \gamma^\rho(\mathbb{R}^n)$.

The following application is a general version of Corollaries 1 and 2 in [7]. Let P be hypoelliptic s.t. $\deg_{x_1} P = \upsilon$ and (4.1) is satisfied for $x' = x_1$.

As we showed in [8] the space of locally slowly growing solutions of $P(D)$ defined on $\mathbb{R}\backslash\{0\} \times \mathbb{R}^{n-1}$ coincides with the space of all solutions f having a υ-fold distributional boundary value on $\mathbb{R}^{n-1}$. This boundary value mapping R^{υ} is a (topological) homomorphism onto the υ-fold copy of $D_b'(\mathbb{R}^{n-1})$.

COROLLARY 4.6. *Let P satisfy the above assumptions. Then for every $\varepsilon > 0$ and every υ-tuple (T_i) of distributions there exists a locally slowly growing solution $u_{(T_i)}$ s.t.* supp $u_{(T_i)} \subset [-\varepsilon, \varepsilon] \times \mathbb{R}^{n-1}$ *and* $R^{\upsilon}(u_{(T_i)}) = (T_i)$.

Proof. By [8] there is a locally slowly growing solution $v_{(T_i)}$ representing (T_i). Now $v_{(T_i)}$ can be cut off as in Corollary 4.4(b) without hurting the growth condition.

In [9] the representation of $\mathscr{S}'$ by slowly growing solutions was proved. These solutions cannot have their supports bounded with respect to x_1 as this would imply the existence of a nonzero solution which is polynomially growing on $\mathbb{R}^n$ and has x_1-bounded support. This contradicts a classical theorem on the uniqueness of the Cauchy problem.

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